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Local stability of Gerstner's waves

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A general method is presented to investigate the hydrodynamic stability of ideal incompressible or barotropic flows described in a Lagrangian representation. Based on the theory of short-wavelength instabilities, the problem is reduced to a transport equation which involves only the distortion matrix of the equilibrium flow. The theory is applied to Gerstner's rotational free-surface gravity waves. It is shown that they are three-dimensionally unstable when their steepness exceeds 1/3.

1. Introduction

The motion of a continuous medium may be described either by the trajectories of its material particles or by the velocity field expressed at any geometric point of space. Although both have been formalized by Euler himself (see Truesdell 1954), these alternative but equivalent descriptions are called respectively Lagrangian and Eulerian. Each has its advantages. The Lagrangian description is more convenient to describe the deformations of the medium and is generally used in elasticity theory. For hydrodynamics the Eulerian approach is clearly preferable, probably because of the extreme complexity of the viscous terms, as outlined by Yakubovich & Zenkovich (2001, 2002).

In ideal flows, both descriptions have similar levels of complexity and may be used to integrate the equations of motion. Explicit solutions are known in each representation but only the simplest ones have an explicit form in both descriptions. Exact Eulerian solutions may be found in Majda & Bertozzi (2002) or Friedlander & Lipton-Lifschitz (2003). Lagrangian solutions are less numerous but present the great advantage that the kinematics of deformations may be described explicitly. The importance of kinematics in fluid dynamics was underlined by Truesdell (1954) who wrote in the preface of his monograph: "a kinematical result is a result valid forever, no matter how time and fashion may change the 'laws' of physics." Indeed, fundamental results such as Kelvin's circulation theorem may be proved without needing Newton's second law, but with the milder assumption that the acceleration is derivable from a potential (Truesdell 1954; Serrin 1959). The close interplay between the Lagrangian description and kinematics led Yakubovich & Zenkovich to reformulate the Euler equations in terms of the distortion (or Jacobi) matrix, yielding new exact solutions. We shall show here that their formulation is almost entirely kinematical and may be easily extended to barotropic ideal flows (§ 2).

Nowadays, studies in hydrodynamic stability use highly sophisticated mathematical and numerical methods, sometimes omitting the underlying physical mechanisms. However, the theory of short-wavelength instabilities developed by Eckhoff & Soresletten (1978), Bayly (1987), Friedlander & Vishik (1991) and Lifschitz & Hameiri (1991) has reconciled hydrodynamic stability with the physics of deformation. This appears clearly in the papers by Bayly (1988), Friedlander & Vishik (1992) and Lifschitz (1994). Here, we shall show that when the equilibrium flow is given in Lagrangian form, the equations governing the evolution of short-wave disturbances are greatly simplified thanks to kinematics and involve only the distortion matrix of the basic flow (§ 3).

This method is applied in § 4 to the rotational finite-amplitude free-surface waves discovered by von Gerstner in 1802 and rediscovered by Rankine in 1863 (see Lamb 1932; Serrin 1959; Kinsman 1965). The stability of this exact solution of the Euler equations has never been investigated perhaps owing to their limited physical relevance as discussed in § 5, but more probably for technical reasons. Indeed, to our knowledge, we present here the first stability analysis of a Lagrangian flow which is not explicit in the Eulerian representation.

2. Kinematics and Lagrangian dynamics

2.1. *Kinematics of deformations*

Let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ be a Cartesian orthonormal basis associated with a Galilean frame of reference, $\boldsymbol{x} = x_i \boldsymbol{e}_i$ (with the convention of summation for repeated indices), the vector position of any geometric point in space. The motion of a continuous medium is either described by its velocity field $\boldsymbol{U}(\boldsymbol{x}, t)$ – the Eulerian description – or by the particle flow trajectories $\boldsymbol{X}(t; \boldsymbol{a})$, where $\boldsymbol{a}$ is a parameter – the Lagrangian label – associated with each particle path. We recall that $\boldsymbol{a}$ is not necessarily the initial position $\boldsymbol{X}_0$ of the fluid particle. We shall assume here that the motion is free from any discontinuity, so that $\boldsymbol{a}$ parametrizes a single trajectory. In the paper, the same notation will be used for any scalar, vector or tensor field f described in either a Eulerian or Lagrangian representation: $f(\boldsymbol{X}(t; \boldsymbol{a}), t) \equiv f(t; \boldsymbol{a}) \equiv f(t)$. The material derivative will be noted $\dot{f}$ and initial data f_0. If $\dot{f} = 0$, f is a Lagrangian invariant. Trajectories are solutions of $\dot{\boldsymbol{X}} = \boldsymbol{U}$.

Deformations of the medium are characterized by the distortion matrix $\mathsf{\boldsymbol{F}}$ with components†

$$F_{ij} = \partial X_i / \partial a_j. \tag{2.1}$$

The continuity assumption requires that the Jacobian $J = \det \mathsf{\boldsymbol{F}}$ of the transformation is such that $0 < |J(t)| < +\infty$ and satisfies $\dot{J} = J \operatorname{div} \boldsymbol{U}$ (Lamb 1932; Serrin 1959). Since J never vanishes, $\mathsf{\boldsymbol{G}} = \mathsf{\boldsymbol{F}}^{-1}$ is defined. $\mathsf{\boldsymbol{F}}$ and $\mathsf{\boldsymbol{G}}$ satisfy respectively

$$\dot{\mathsf{\boldsymbol{F}}} = \mathsf{\boldsymbol{L}}\mathsf{\boldsymbol{F}}, \quad \dot{\mathsf{\boldsymbol{G}}} = -\mathsf{\boldsymbol{G}}\mathsf{\boldsymbol{L}}, \tag{2.2a, b}$$

where $\mathsf{\boldsymbol{L}}$ is the velocity gradient tensor with components $L_{ij} = \partial U_i / \partial x_j$.

From (2.2), the solutions of the equations

$$\dot{\boldsymbol{\eta}} = \mathsf{\boldsymbol{L}}\boldsymbol{\eta}, \quad \dot{\boldsymbol{\xi}} = -\mathsf{\boldsymbol{L}}^T \boldsymbol{\xi}, \tag{2.3a, b}$$

where $\mathsf{\boldsymbol{L}}^T$ is the transpose of $\mathsf{\boldsymbol{L}}$, are respectively

$$\boldsymbol{\eta}(t) = \mathsf{\boldsymbol{F}}(t)\mathsf{\boldsymbol{G}}_0\boldsymbol{\eta}_0, \quad \boldsymbol{\xi}(t) = \mathsf{\boldsymbol{G}}^T(t)\mathsf{\boldsymbol{F}}_0^T\boldsymbol{\xi}_0. \tag{2.4a, b}$$

As is well known, (2.3*a*) governs the evolution of an infinitesimal frozen-in material vector $\delta \boldsymbol{X}$, whereas (2.3*b*) is the equation for an oriented surface element $\delta \boldsymbol{X}' \times \delta \boldsymbol{X}''$

† As pointed out by an anonymous referee, the tensorial character of $\mathsf{\boldsymbol{F}}$ is not generally true when $\boldsymbol{a} \neq \boldsymbol{X}_0$ and a_j represent some curvilinear coordinates, while X_i are Cartesian. The terminology 'distortion matrix' is borrowed from Yudovich (2000).

in an incompressible medium (Truesdell 1954). Lifschitz (1994) points out that if the motion is steady (in the Eulerian sense), then the velocity field $\boldsymbol{U}$ satisfies (2.3a). Finally the following kinematical relation may be established from (2.2):

$$\dot{\boldsymbol{F}}^T\boldsymbol{F} - \boldsymbol{F}^T\dot{\boldsymbol{F}} = -2\boldsymbol{F}^T\boldsymbol{A}\boldsymbol{F}, \tag{2.5}$$

where $\boldsymbol{A} = \frac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^T)$ is the skew-symmetric vorticity tensor, related to the vorticity $\boldsymbol{\Omega} = \operatorname{curl} \boldsymbol{U}$ of the flow through the relation $A_{ij} = -\frac{1}{2}\epsilon_{ijk}\Omega_k$.

2.2. *Circulation-preserving motions*

So far, purely kinematical considerations have been developed. The generality of the preceding discussion is now restricted by assuming that the acceleration derives from a potential: $\dot{\boldsymbol{U}} = -\nabla\Phi$. This property holds for any inviscid incompressible or barotropic flow subjected to conservative body forces. Under this hypothesis, Kelvin's circulation theorem is satisfied, so that such motions are said to be circulation-preserving (Truesdell 1954; Serrin 1959). In that case $J(t)\boldsymbol{\Omega}(t)$ is governed by (2.3a) so that from (2.4a)

$$J(t)\boldsymbol{G}(t)\boldsymbol{\Omega}(t) = J_0\boldsymbol{G}_0\boldsymbol{\Omega}_0. \tag{2.6}$$

This is Cauchy's invariant. In an incompressible medium when the Lagrangian label $\boldsymbol{a}$ coincides with the initial position $\boldsymbol{X}_0$ of the fluid particle, (2.6) reduces to the well-known relation: $\boldsymbol{\Omega}(t) = \boldsymbol{F}(t)\boldsymbol{\Omega}_0$. Indeed if $\boldsymbol{a} = \boldsymbol{X}_0$, $\boldsymbol{F}_0 = \boldsymbol{G}_0 = \boldsymbol{I}$ and $J_0 = 1$.

For circulation-preserving motions, the vorticity tensor $\boldsymbol{A}$ obeys†

$$\dot{\boldsymbol{A}} + \boldsymbol{S}\boldsymbol{A} + \boldsymbol{A}\boldsymbol{S} = 0, \tag{2.7}$$

where $\boldsymbol{S} = \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T)$ is the symmetric rate-of-deformation tensor. As a consequence, from (2.2a) and (2.7), it may be checked that the matrix $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{A}\boldsymbol{F}$ is a Lagrangian invariant, so that the right-hand side of (2.5) is invariant as established in the incompressible case by Yakubovich & Zenkovich (2001). $\boldsymbol{C}$ being skew-symmetric, the three independent components are related to three scalar quantities C_k defined by $C_{ij} = -\frac{1}{2}\epsilon_{ijk}C_k$. Clearly the C_k are invariant and coincide in fact with the three components of Cauchy's invariant (2.6). This was pointed out by Yakubovich & Zenkovich in the incompressible case, but holds for any circulation-preserving motion. We finally note that $C_k = \epsilon_{klm}\partial(F_{nm}U_n)/\partial a_l$, known as Eckart's invariants (Salmon 1988).

2.3. *Lagrangian dynamics of inviscid flows*

In the Eulerian representation, the dynamics of an ideal incompressible flow is described by the velocity and pressure fields $\boldsymbol{U}(\boldsymbol{x}, t)$ and $P(\boldsymbol{x}, t)$. In the Lagrangian form the variables are usually $\boldsymbol{X}(t; \boldsymbol{a})$ and $P(t; \boldsymbol{a})$, solutions of Euler equations $\boldsymbol{F}^T\ddot{\boldsymbol{X}} = -\nabla_{\boldsymbol{a}}\Phi$ where $\Phi = P/\rho$: plus the potential of the external forces (Serrin 1959). Incompressibility requires that $\det \boldsymbol{F}(t) = \det \boldsymbol{F}_0$. Several solutions are known in the Lagrangian representation, such as Gerstner's waves and related solutions (Pollard 1970; Constantin 2001), or the so-called Ptolemaic flows (Abrashkin & Yakubovich 1984). Lagrangian flows exhibiting finite-time blow-up have been constructed by Stuart (1987, 1998) and Childress *et al.* (1989). These Lagrangian solutions cannot be explicitly found in the Eulerian representation.

Alternatively Yakubovich & Zenkovich (2001) discovered that the Lagrangian motion of an incompressible ideal fluid subjected to conservative body forces may

† The proof is similar to the incompressible case described for instance in Yudovich (2000).

be described completely by the distortion matrix $F_{ij}=\partial X_i/\partial a_j$. Its evolution is governed by the kinematical relation (2.5). Together with incompressibility constraint $\det \boldsymbol{F}(t)=\det \boldsymbol{F}_0$ and the consistency relations $\partial F_{ij}/\partial a_k=\partial F_{ik}/\partial a_j$, this constitutes a closed set of equations equivalent to the Euler equations. Kinematics is therefore an essential ingredient of this approach. It led to the discovery of several three-dimensional unsteady exact solutions such as precessing or stretched vortices, flows with curvilinear vortex lines (Yakubovich & Zenkovich 2001, 2002) that have no counterpart in the Eulerian description.

We conclude this discussion by noting that this new Lagrangian formulation may be easily extended to barotropic flows, i.e. compressible flows with state law $P=P(\rho)$, such as a perfect gas in homentropic evolution (Serrin 1959). $\boldsymbol{F}(t)$ is governed by the same system of equations, except the incompressibility constraint which is now replaced by $\dot{J}=J\,\mathrm{div}\,\boldsymbol{U}$ where $J=\det \boldsymbol{F}$. Velocity divergence may be expressed as a function of $\boldsymbol{F}$ as follows: $\mathrm{div}\,\boldsymbol{U}=\mathrm{tr}\,\boldsymbol{L}=\mathrm{tr}(\dot{\boldsymbol{F}}\boldsymbol{G})$. Once the problem solved for $\boldsymbol{F}$, the density field may be deduced by the usual relation $J(t)\rho(t)=J_0\rho_0$ (Lamb 1932).

3. The theory of short-wavelength instabilities

3.1. *Eulerian equilibrium flows*

Now let $\boldsymbol{U}$ be the velocity field of an ideal incompressible flow subjected to conservative body forces. The linear stability of this equilibrium (or basic) flow is characterized by the growth or decay of infinitesimal disturbances $\boldsymbol{u}$ governed by the linearized Euler equations. The theory of short-wave instabilities consists of considering the evolution of a rapidly varying WKB wave packet $\boldsymbol{u}=\boldsymbol{v}\mathrm{e}^{\mathrm{i}\phi/\varepsilon}$ with $\varepsilon \ll 1$ (see details in Lifschitz & Hameiri 1991). By defining the wave vector $\boldsymbol{\xi}=\nabla\phi$, it may be shown that the stability problem is reduced to a system of ordinary differential equations that evolves along the trajectories of the equilibrium flow:

$$\dot{\boldsymbol{X}}=\boldsymbol{U}, \quad \dot{\boldsymbol{\xi}}=-\boldsymbol{L}^T\boldsymbol{\xi}, \quad \dot{\boldsymbol{v}}=(2\boldsymbol{\xi}\otimes\boldsymbol{\xi}/|\boldsymbol{\xi}|^2-\boldsymbol{I})\boldsymbol{L}\boldsymbol{v}. \qquad (3.1a, b, c)$$

Here $\boldsymbol{L}$ is the basic velocity gradient tensor, $\boldsymbol{\xi}\otimes\boldsymbol{\xi}$ the tensor with components $\xi_i\xi_j$, and $\boldsymbol{I}$ the identity matrix. This system is completed by initial data for the wave vector and the velocity amplitude: $|\boldsymbol{\xi}_0|=|\boldsymbol{v}_0|=1$ and $\boldsymbol{\xi}_0 \perp \boldsymbol{v}_0$.

This elegant set of equations was derived independently by Bayly (1987), Friedlander & Vishik (1991), and Lifschitz & Hameiri (1991). The asymptotic behaviour of the velocity amplitude $\boldsymbol{v}$ characterizes stability: It is proved that *the equilibrium flow (steady or not) is unstable if there exists at least one trajectory along which* $|\boldsymbol{v}(t)|$ *grows unboundedly* (Lifschitz & Hameiri 1991). One talks about local instabilities because they are localized along the particle paths. Furthermore, for steady flows, exponential growth gives information on spectral bounds of the associated linearized operators (see details in Friedlander & Lipton-Lifschitz 2003).

This method has always been used for equilibrium flows described in a Eulerian form so that the trajectory solutions of (3.1*a*) are in general not known explicitly. Therefore the kinematical solution (2.4*b*) of (3.1*b*) cannot be exploited. In many circumstances however, (3.1) may be locally integrated along particular trajectories, and various classical stability results have been recovered and generalized: Rayleigh and Leibovich–Stewartson criteria (Eckhoff & Storesletten 1978; Bayly 1987, 1988; Sipp & Jacquin 2000), hyperbolic and elliptical instabilities (Friedlander & Vishik 1991; Lifschitz & Hameiri 1991; Leblanc 1997; Le Dizès 2000), etc. New criteria for steady flows were derived in Friedlander & Vishik (1992) and Lifschitz (1994), and instability mechanisms in unsteady flows were discovered in Bayly, Holm & Lifschitz

(1996) and Leblanc (2000). A complete list of references is given in the survey article by Friedlander & Lipton-Lifschitz (2003).

Finally it is worth noting that instead of solving (3.1*c*) for the velocity amplitude, an equation for the vorticity disturbance $\boldsymbol{\omega}=\boldsymbol{\xi}\times\boldsymbol{v}$ may be constructed (Lifschitz 1994):

$$\dot{\boldsymbol{\omega}} = \boldsymbol{L}\boldsymbol{\omega} + \boldsymbol{\omega}\times\boldsymbol{\xi}(\boldsymbol{\Omega}\cdot\boldsymbol{\xi})/|\boldsymbol{\xi}|^2, \tag{3.2}$$

where $\boldsymbol{\Omega}$ is the vorticity of the basic flow. An unbounded solution of (3.2) along a given trajectory, such that $|\boldsymbol{\xi}_0|=|\boldsymbol{\omega}_0|=1$ and $\boldsymbol{\xi}_0\perp\boldsymbol{\omega}_0$ at the initial time, guarantees instability in the vorticity norm. Along an irrotational trajectory $\boldsymbol{\omega}$ satisfies (2.3*a*).

3.2. *Lagrangian equilibrium flows*

As we shall see now, the theory of short-wave instabilities is greatly simplified when the equilibrium flow is described in Lagrangian form. Indeed, the trajectories being known explicitly, (3.1*a*) is already solved. The distortion matrix $\boldsymbol{F}$ defined in (2.1), its inverse $\boldsymbol{G}$, their time derivative and their initial value may then be computed by differentiation. Therefore the solution of (3.1*b*) given in (2.4*b*) is also explicit. Furthermore the basic velocity gradient $\boldsymbol{L}$ involved in the amplitude equation (3.1*c*) may also be calculated thanks to the kinematical relations (2.2). As a consequence, the local stability of an inviscid incompressible Lagrangian equilibrium flow is reduced to the study of the following transport equation:

$$\dot{\boldsymbol{v}} = \left(\boldsymbol{F} - 2\boldsymbol{G}^T\frac{\boldsymbol{n}_0\otimes\boldsymbol{n}_0}{\boldsymbol{n}_0^T\boldsymbol{G}\boldsymbol{G}^T\boldsymbol{n}_0}\right)\dot{\boldsymbol{G}}\boldsymbol{v}, \quad \boldsymbol{n}_0 = \boldsymbol{F}_0^T\boldsymbol{\xi}_0, \tag{3.3}$$

with initial data satisfying $|\boldsymbol{\xi}_0|=|\boldsymbol{v}_0|=1$ and $\boldsymbol{\xi}_0\perp\boldsymbol{v}_0$.

The solution of this equation depends parametrically on the Lagrangian label $\boldsymbol{a}$ associated with each trajectory, and on the orientation of the initial wave vector $\boldsymbol{\xi}_0$ pointing to the unit sphere. But the unbounded growth of a particular solution of (3.3) is sufficient to prove instability. Equation (3.2) for the vorticity amplitude may also be transformed in a similar fashion. Indeed, the flow being incompressible, it may be checked from (2.4*b*) and (2.6) that $\boldsymbol{\Omega}(t)\cdot\boldsymbol{\xi}(t)=\boldsymbol{\Omega}_0\cdot\boldsymbol{\xi}_0$, as initially noted by Lifschitz (1994).

Only slight modifications are needed to describe the local instabilities of a barotropic ideal flow. For example, in the case of a perfect gas in homentropic evolution (Leblanc 2001), it has been shown that the stability problem still consists of system (3.1), where $\frac{1}{2}(\mathrm{tr}\,\boldsymbol{L})\boldsymbol{v}$ is added to the right-hand side of (3.1*c*). In the Lagrangian representation, this additive term is also explicit thanks to (2.2).

To conclude, the present approach may be applied to any ideal incompressible or barotropic flow described in the Lagrangian representation. It is worth noting that the only information required on the basic flow is its distortion matrix $\boldsymbol{F}$, as illustrated in (3.3). Therefore the theory is particularly well suited to the approach of Yakubovich & Zenkovich.

4. Stability of Gerstner's waves

4.1. *Description of the flow*

The theory is now applied to Gerstner's waves, an exact solution of the Euler equations. In the Galilean frame $(O;\boldsymbol{i},\boldsymbol{j},\boldsymbol{k})$, let ρ be the density of an incompressible inviscid free-surface flow, subject to gravity $-g\boldsymbol{j}$. The particle paths in Gerstner's

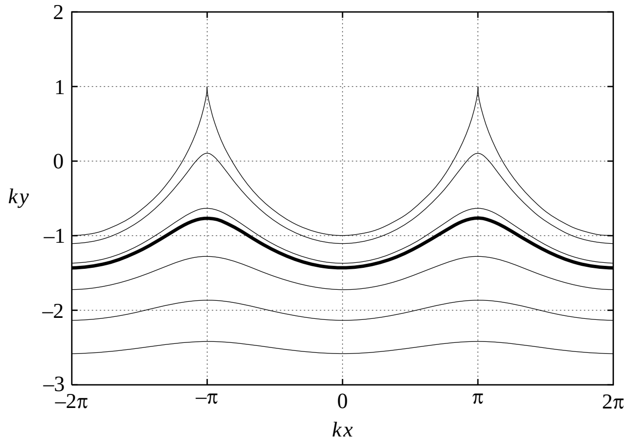

FIGURE 1. Contour lines of pressure (or vorticity) of Gerstner's waves at $t = 0$, plotted for $kb = -2.5$ (bottom curve) to $kb = 0$ (top curve) with increment 0.5. Any level of constant pressure may be considered as the free surface of the flow. The thick line corresponds to $kb = -\ln 3$ above which Gerstner's waves are unstable to short-wavelength disturbances.

waves are given parametrically by (Lamb 1932; Serrin 1959; Kinsman 1965):

$$X(t) = a + k^{-1}\mathrm{e}^{kb}\sin(ka - \omega t), \quad Y(t) = b - k^{-1}\mathrm{e}^{kb}\cos(ka - \omega t), \quad Z(t) = c,$$

where $k > 0$ is the spatial wavenumber, $\omega = \sqrt{gk}$ the frequency, and $\boldsymbol{a} = (a, b, c)$ is the Lagrangian label associated with each trajectory. Its physical meaning is made clear by noting that $(X - a)^2 + (Y - b)^2 = \mathrm{e}^{2kb}/k^2$, so that fluid particles rotate clockwise in planes $Z = \text{const}$ along circles of radius e^{kb}/k and centre (a, b). Continuity and incompressibility are ensured if $b < 0$ (Serrin 1959).

The pressure field is given by $P(b) = -\rho g b + \frac{1}{2}\rho\omega^2\mathrm{e}^{2kb}/k^2 + \text{const}$. Surfaces of constant pressure are parametrized by b. Any one, say b_0, may be chosen as free surface for the flow. The additive constant may be adjusted to get a prescribed value P_0 on the free surface. Levels of constant pressure are represented on figure 1. They are trochoids with crest–trough amplitude $2\mathrm{e}^{kb}/k$. The steepness parameter of the free-surface profile is e^{kb_0} (half-amplitude multiplied by wavenumber). When time varies, those gravity waves propagate from left to right with celerity ω/k. The limiting (although unphysical) case $b_0 = 0$ corresponds to a cycloidal free surface with sharp crests. When $b_0 \to -\infty$, linear gravity waves are recovered. For weak amplitude it has been known since Rayleigh that the free surface of irrotational Stokes waves coincides with a trochoid up to third order in the amplitude parameter (Lamb 1932; Kinsman 1965). However, contrary to Stokes waves, Gerstner's waves are rotational with vorticity $\Omega(b) = 2\omega\mathrm{e}^{2kb}/(1 - \mathrm{e}^{2kb})$. Vorticity decreases rapidly with depth and is infinite when $b = 0$. Accordingly, levels of constant pressure and vorticity coincide.

It is convenient to introduce dimensionless variables $\boldsymbol{X}^* = k\boldsymbol{X}$, $\boldsymbol{a}^* = k\boldsymbol{a}$, and $t^* = \omega t$. Trajectories are now given by, omitting the stars,

$$X(t) = a + \mathrm{e}^{b}\sin(a - t), \quad Y(t) = b - \mathrm{e}^{b}\cos(a - t), \quad Z(t) = c.$$

The dimensionless distortion matrix $\boldsymbol{F}$ defined in (2.1) is

$$\boldsymbol{F}(t) = \begin{pmatrix} 1 + \delta\cos(a - t) & \delta\sin(a - t) & 0 \\ \delta\sin(a - t) & 1 - \delta\cos(a - t) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.1}$$

where $\delta = e^b$ $(0 \leqslant \delta < 1)$. With $J = \det \boldsymbol{F} = 1 - \delta^2$, the inverse of $\boldsymbol{F}$ is

$$\boldsymbol{G}(t) = \frac{1}{J}\begin{pmatrix} 1-\delta\cos(a-t) & -\delta\sin(a-t) & 0 \\ -\delta\sin(a-t) & 1+\delta\cos(a-t) & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4.2}$$

The equilibrium velocity gradient $\boldsymbol{L}$ expressed in the Lagrangian representation may then be calculated using (2.2). It may be written as (in agreement with Serrin 1959)

$$\boldsymbol{L}(t) = \frac{\delta}{J}\begin{pmatrix} \sin(a-t) & -\cos(a-t) & 0 \\ -\cos(a-t) & -\sin(a-t) & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{\delta^2}{J}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.3}$$

from which we deduce the dimensionless vorticity $\Omega = 2\delta^2/J$. The symmetric part of (4.3) shows that each particle experiences locally, along its circular path, a strain field of intensity δ/J which rotates itself with dimensionless angular velocity $-1/2$.

4.2. *Stability analysis*

It remains to solve (3.3) to characterize the local stability of Gerstner's solution. Although the matrix coefficient involved in the right-hand side may now be computed explicitly, integration of (3.3) requires numerics. However, the equilibrium flow being two-dimensional in the plane $(O; \boldsymbol{i}, \boldsymbol{j})$, the class of disturbances characterized by $\boldsymbol{\xi}_0 = \boldsymbol{k}$ is of particular interest. Various instability criteria have been formulated with such three-dimensional perturbations (Bayly 1988; Leblanc 1997; Sipp & Jacquin 2000).

From (2.4*b*), (4.1) and (4.2), we get $\boldsymbol{\xi}(t) = \boldsymbol{k}$, and (3.3) becomes simply

$$\dot{\boldsymbol{v}} = \boldsymbol{F}\dot{\boldsymbol{G}}\boldsymbol{v} = -\boldsymbol{L}\boldsymbol{v}. \tag{4.4}$$

This shows that the velocity disturbance lies in the plane $(O; \boldsymbol{i}, \boldsymbol{j})$ for all time. Unlike (2.3) an explicit solution of (4.4) seems not to be known in the general case. Fortunately, for Gerstner's waves integration is possible. Indeed (4.3) being a rotating strain field, the velocity gradient becomes time-independent in a frame which rotates locally with the strain. Then we introduce a rotating basis $(\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k})$ defined by

$$\boldsymbol{i}'(t) = \boldsymbol{i}\cos(-t/2) + \boldsymbol{j}\sin(-t/2), \quad \boldsymbol{j}'(t) = -\boldsymbol{i}\sin(-t/2) + \boldsymbol{j}\cos(-t/2).$$

With a slight abuse of notation, let $\boldsymbol{v}'$ and $\boldsymbol{L}'$ be respectively the vector and matrix of the components of $\boldsymbol{v}$ and $\boldsymbol{L}$ expressed in $(\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k})$. Introducing the change-of-basis matrix

$$\boldsymbol{P}(t) = \begin{pmatrix} \cos(-t/2) & -\sin(-t/2) & 0 \\ \sin(-t/2) & \cos(-t/2) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{P}^{-1} = \boldsymbol{P}^T,$$

$\boldsymbol{v}'$ and $\boldsymbol{L}'$ are defined by the usual relations $\boldsymbol{v} = \boldsymbol{P}\boldsymbol{v}'$ and $\boldsymbol{L} = \boldsymbol{P}\boldsymbol{L}'\boldsymbol{P}^T$. Noting that

$$\dot{\boldsymbol{P}} = \boldsymbol{R}\boldsymbol{P}, \quad \boldsymbol{R} = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{L}' = \frac{\delta}{J}\begin{pmatrix} \sin a & -\delta-\cos a & 0 \\ \delta-\cos a & -\sin a & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and that $\boldsymbol{P}^T\boldsymbol{R}\boldsymbol{P} = \boldsymbol{R}$, it is easy to prove that, in the rotating basis, (4.4) is

$$\dot{\boldsymbol{v}}' = -(\boldsymbol{L}' + \boldsymbol{R})\boldsymbol{v}',$$

which is as expected an autonomous system.

The eigenvalues of $\boldsymbol{L}'+\boldsymbol{R}$ therefore determines the behaviour of both $\boldsymbol{v}'$ and $\boldsymbol{v}=\boldsymbol{P}\boldsymbol{v}'$ since $\boldsymbol{P}$ is time-periodic. Eigenvalues satisfy the characteristic equation

$$\lambda^2 = \frac{1}{4}\left(\frac{9\delta^2-1}{1-\delta^2}\right), \tag{4.5}$$

so that exponential growth occurs if and only if $\delta > 1/3$ since $0 \leqslant \delta < 1$, that is when the dimensionless vorticity $\Omega = 2\delta^2/(1-\delta^2)$ exceeds 1/4. The temporal growth rate of the short-wavelength disturbances is therefore the positive root in (4.5). Returning to dimensional variables, the following theorem has been proved:

Theorem. *Gerstner's waves with frequency ω are unstable when the free-surface vorticity exceeds $\omega/4$, or equivalently when the steepness parameter exceeds 1/3. Instabilities are localized in the layer defined by $-\ln 3 < kb \leqslant kb_0$, where $b_0 < 0$ parametrizes the free surface. Their growth rate is*

$$\lambda(b) = \frac{\omega}{2}\left(\frac{9\mathrm{e}^{2kb}-1}{1-\mathrm{e}^{2kb}}\right)^{1/2}.$$

The dependence of the growth rate on the Lagrangian label b shows that levels of constant pressure, vorticity and growth rates coincide. The critical level $kb = -\ln 3$ has been plotted on figure 1. Above this threshold, $\lambda(b)$ grows with the steepness. When the free surface is cycloidal the growth rate is infinite.

Our analysis has been restricted to local disturbances with wave vector $\boldsymbol{\xi}(t) = \boldsymbol{k}$. Numerical computations of (3.3) for all other possible orientations of the initial wave vector $\boldsymbol{\xi}_0$ on the unit sphere have been carried out. The results are not reported here because they did not reveal any new features. In particular, no instability has been found when $kb \leqslant -\ln 3$.

We remark finally that the instability mechanism found above is purely rotational. Indeed if the equilibrium flow is irrotational ($\boldsymbol{\Omega} = 0$), the equation governing the vorticity disturbance (3.2) simplifies to (2.3*a*), the solution of which is given in (2.4*a*). It is known that if the flow is two-dimensional and steady, such as Stokes waves in a co-moving frame, the explicit solution (2.4*a*) cannot grow faster than algebraically except on hyperbolic stagnation points (see Godeferd, Cambon & Leblanc 2001). Therefore in Stokes waves, local instabilities cannot grow exponentially. Thus this mechanism differs from those discovered in irrotational free-surface gravity waves (Dias & Kharif 1999).

5. Discussion

The theory of short-wavelength instabilities appears to be an efficient tool to study the hydrodynamic stability of Lagrangian inviscid flows. As explained in the paper, the reason is that the formulation of the problem involves only the distortion matrix which is explicit when the equilibrium flow is described in the Lagrangian representation.

The method has been applied to Gerstner's exact solution which has been shown to be unstable when the wave profile is steep enough. One may wonder why the stability of Gerstner's waves has not been addressed in the literature since 1802? Two reasons may be put forward: physical and technical. Gerstner's waves are indeed of limited physical relevance because they are rotational. As a consequence they cannot be created from a state of rest thanks to the Lagrange theorem, nor by pressure forces (Lamb 1932). However, we can imagine that a train of Stokes waves propagating on the sea surface and crossing a localized region of vorticity could locally be described by Gerstner's solution. Although such rotational waves have never been observed experimentally, the simplicity of Gerstner's solution is useful for academic studies (Naciri & Mei 1992; Fuks & Voronovich 2002). At any rate, realistic or not, Gerstner's waves if they exist cannot be too steep otherwise they are unstable. The critical steepness is 1/3. Above that value, the trochoidal profile of Gerstner's solution is close to Stokes waves (see Naciri & Mei 1992). Therefore, experimental observation of Gerstner's waves seems even more difficult.

The second reason why the stability of Lagrangian solutions has not been addressed in the past is technical. Indeed, conventional methods used in hydrodynamic stability usually consider equilibrium flows in which velocity and pressure fields are explicit in the Eulerian representation. If this is not the case the analysis becomes extremely difficult. Gerstner's waves are a preliminary application of the method we have presented. Stability analyses of other Lagrangian vortical flows such as Ptolemaic vortices (Abrashkin & Yakubovich 1984) including the combined effects of density stratification and planetary rotation are under current investigation. Study of more complex vortex flows (Yakubovich & Zenkovich 2001, 2002) is left for future work. Finally, the stability of exact inviscid solutions exhibiting finite-time blow-up (Stuart 1987, 1998; Childress *et al.* 1989) might be of great interest to understand the behaviour of singular solutions of Euler equations, if they exist (Majda & Bertozzi 2002).

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